Decoherence and classical predictability of phase space histories.

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Abstract

We consider the decoherence of phase space histories in quantum brownian motion models, consisting of a particle moving under a potential $V(x)$ in contact with a heat bath of temperature T and dissipation constant γ in the Markovian regime. The evolution of the density operator for this system is thus described by a non-unitary master equation. The phase space histories are described by quasiprojectors consisting of gaussian density matrices smeared over large phase space cells. They are characterized by the size $[\Gamma]$ of the phase space cell together with the size $[M]$ of the margin (the region at the boundary of Γ in which the Weyl symbol of the projector goes from 1 to 0). By generalizing an earlier result of Hagedorn on the unitary evolution of coherent states, we show that an initial Gaussian density matrix remains approximately Gaussian under non-unitary time evolution, and derive a bound giving the validity of this approximation. This result is then used, following earlier work of Omnès to compute the time evolution of the phase space projectors under the master equation: The evolution of an initial projector P onto a cell Γ is approximately equal to another phase space projector P' onto a cell Γ', where Γ'

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is the classical dissipative evolution of Γ. Furthermore, the expected unpredictability due to thermal fluctuations is reflected in the fact that the margin of P' (and hence the effective region it occupies in phase space) is greater than that of P. We thus show that histories of phase space samplings approximately decohere, and that the probabilities for these histories are peaked about classical dissipative evolution, but with an element of unpredictability due to the noise produced by the environment.

1 Introduction

Ever since the first days of quantum mechanics, the question of the emergence of the classical deterministic world from the underlyinfg probabilistic laws is considered of particular importance. It is closely connected to the measurement problem, since any measurement involves the interaction of the system under observation with a large deterministic device.

Different aspects of this problem have been adressed extensively in the literature. Still, there are issues that remain unclear. Among them are question of the validity of the description of many body systems with collective variables that evolve under quasiclassical equations of motion, the conditions under which classical predictability arises in a quantum system, its possible breakdown at long times and how it is affected by environmentally induced noise. In this paper we will address these problems, as appearing in a wide class of open quantum systems, namely quantum Brownian motion models.

Our analysis lies within the framework of the decoherent histories approach to quantum mechanics, as set out by Griffiths [1],Omnès [2, 3, 4] and Gell-Mann and Hartle [5]. The main element of this formalism is the notion of history. A history is a string of projection operators at consecutive moments of time. We can build sets of histories, by taking a partition of the unit operators into projectors $P_\alpha(t)$ at each moment of time. These operators satisfy:

$$P_\alpha P_\beta = \delta_{\alpha\beta} P_\alpha \tag{1.1}$$

$$\sum_\alpha P_\alpha = 1 \tag{1.2}$$

We usually consider them to evolve in the Heisenberg picture

$$P_\alpha(t) = e^{iHt} P_\alpha e^{-iHt} \tag{1.3}$$

An element of a set of such histories reads :

$$C_\alpha = P_{\alpha_n}^n(t_n) \dots P_{\alpha_2}^2(t_2) P_{\alpha_1}^1(t_1) \tag{1.4}$$

A space of histories has a natural candidate probability measure

$$p(C_\alpha) = Tr(C_\alpha \rho_0 C_\alpha) \tag{1.5}$$

This measure can be interpreted as a probability only if the standard probability sum rules are satisfied. The set of histories is then called consistent (or decoherent). The condition for this can be writen in terms of the decoherence functional , a complex valued function of pairs of histories:

$$D(\alpha, \alpha') = Tr(C_\alpha \rho_0 C_{\alpha'}) \qquad (1.6)$$

The necessary and sufficient condition for the sum rules to be satisfied is then

$$ReD(\alpha, \alpha') = 0 \qquad \alpha \neq \alpha' \qquad (1.7)$$

We usually employ the stricter condition

$$D(\alpha, \alpha') = 0 \qquad \alpha \neq \alpha' \qquad (1.8)$$

The terms consistency and decoherence of the set of histories are given in the literature to the conditions (1.7) and (1.8) respectively. That assignment of a probability measure to a set of histories, allows us to reason with the histories using the rules of classical logic.

In general, decoherence appears in physical systems, only for coarse grained histories. A typical example of coarse graining is to ignore a number of degrees of freedom of the system, or take projectors into large intervals of position or momentum. The decoherence is in almost all physical situations only approximate, that is, the probability sum rules are satisfied only within an order of ϵ, where ϵ is a small positive number.

In quantum mechanics the most general proposition for a physical system at a given instant of time is represented by a projection operator. In classical mechanics the corresponding entity is the phase space cell. Thus, when we want to discuss classical deterministic behaviour, we have to construct entities that bridge the gap between the two formalisms.

This is done through the introduction of quasiprojectors, that is, approximate projectors onto phase space cells. They are constructed in such a way, that their corresponding symbol in the Wigner representation is a smoothened characteristic function onto some phase space cell. Omnès [3] has shown that histories corresponding to classical hamiltonian evolution of sufficiently large and regular phase space cells, approximately satisfy the consistency conditions. Also the conditional probability for our system to be found in the cell Γ_2 at time t_2, if at time t_1 it was within the cell Γ_1, is

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almost unity, provided Γ_2 is obtained from Γ_1 through evolution according to the classical equations of motion.

In this paper, we are interested in generalizing these results for a class of open systems, namely quantum Brownian motion models. These consist of a particle moving under a potential $V(x)$ and coupled to a large system which is taken as the environment. Usually we take as an environment a collection of harmonic oscillators in a thermal state. By tracing out the environment degrees of freedom, we obtain an effective evolution equation for the density matrix of the particle. The coupling of the environment to the particle is contained within two effects: dissipation and diffusion.

We concentrate on a particular case of those models, namely when the evolution equations are Markovian. This corresponds to the case of ohmic environment at high temperature. This is mainly due to the fact that only in the Markovian limit can we write the decoherence functional in terms of the reduced density matrix.

These models can be used as prototypes for more complicated systems, as for instance particle detectors in early universe or continuous measurement. What is more interesting for our approach, is that we can model this way the effective behaviour of collective or hydrodynamic variables. We can identify the degrees of freedom of the particle with distinguished variables in a many-body system, like for instance the center of mass position and momentum of a macroscopic body, while the environment represents all the ignored degrees of freedom like the positions and momenta of its constituents.

The issue we are concentrating on, is the conditions under which the description of the system with the classical dissipative equations of motion can be accurate. Essentially we want to prove approximate determinism for quantum dissipative systems. Our approach is rather different from other authors that have discussed decoherence of phase space histories [6, 7] or configuration space histories [8] and closer to the spirit of Omnès [3, 4] .

Our main results can be summarized, as follows: We establish the degree of the validity of the gaussian approximation for the Markovian regime of quantum Brownian motion models. We show that a gaussian density matrix remains approximately gaussian centered around the classical path for a large class of potentials $V(x)$. The validity of the approximation is quite good as long as the spread in position is much smaller than the scale on which the non-linearities in the potential become significant.

We use this result to show, that quasiprojectors onto sufficiently large and smooth phase space cells evolve according to the classical equations of motion. The environment induces a degradation on the evolved projectors, which leads essentially to a loss of predictability.

Finally, we construct the decoherence functional in the Markovian regime and establish that histories corresponding to classical evolution on phase space approximately satisfy the consistency condition, and thus approximate determinism arises.

This paper is structured as follows: After giving a brief review of quantum Brownian motion models, we proceed to discuss phase space projectors in section 3. We generalize the construction of quasiprojectors from coherent states of Omnès, using gaussian density matrices. In section 4 we state a theorem on the degree of validity of the gaussian approximation in our models, using which we establish how the quasiprojectors we constructed evolve in Section 5. We construct histories corresponding to classical evolution in Section 6 and discuss their decoherence properties. In the last section we summarize and discuss our results.

The notation we will use is as follows: Whenever there is a possibility of confusion the reduced density matrix will be denoted as $\tilde{\rho}$, otherwise plainly as ρ. The Hilbert space of the wave functions will be denoted as H_ψ and the one of the density matrices under the Hilbert-Schmidt inner product as H_ρ. We write the inner product in H_ψ using the Dirac notation $\langle \psi_1 | \psi_2 \rangle$ and in H_ρ as (ρ_1, ρ_2). The action of an operator $\mathcal{L}$ on an element of H_ρ will be denoted as $\mathcal{L}[\rho]$. The volume of a phase space cell Γ is written $[\Gamma]$. In Section 4 and in the Appendix we will use units such that $\hbar$ is dimensionless.

2   Quantum Brownian Motion Models

We first give the basic features of the model, within which we are going to discuss phase space decoherence. We consider a particle of mass M moving under the influence of a potential $V(x)$ and in contact with a heat bath of harmonic oscillators [9, 10, 11].

The total action of the system reads :

$$S_{tot}[x(t), q_n(t)] = \int dt[\frac{1}{2}M\dot{x}^2 - V(x)] + \sum_n \int dt[\frac{1}{2}m_n\dot{q}_n{}^2$$

$$-\frac{1}{2}m_n\omega_n^2 q_n^2 - C_n q_n x] \qquad (2.1)$$

We concentrate in the behaviour of the distinguished particle and thus construct the reduced density matrix:

$$\tilde{\rho}_t(x, x') = \prod_n \int dq_n \rho_t(x, q_n, x', q_n)$$

We can describe its time evolution using the reduced density matrix propagator J, defined by the relation:

$$\tilde{\rho}_t(x, y) = \int dx_0 dy_0 \quad J(x, y, t | x_0, y_0, 0) \tilde{\rho}(x_0, y_0)$$

Under the assumption that the initial density matrix ρ_0 factorizes, the propagator is given by the path integral expression :

$$J(x_f, y_f, t \mid x_0, y_0, 0) = \int Dx Dy \exp(\frac{i}{\hbar}S[x] - \frac{i}{\hbar}S[y] + \frac{i}{\hbar}W[x, y]) \qquad (2.2)$$

where

$$S[x] = \int_0^t dt(\frac{1}{2}M\dot{x}^2 - V(x))$$

$W[x(t), y(t)]$ is the Feynman Vernon influence functional phase

$$W[x(t), y(t)] = -\int_0^t ds \int_0^s ds'[x(s) - y(s)]\eta(s - s')[x(s') + y(s')]$$
$$+i\int_0^t ds \int_0^s ds'[x(s) - y(s)]\nu(s - s')[x(s') - y(s')] \qquad (2.3)$$

The path integration is taken over all paths satisfying $x(0) = x_0$, $x(t) = x_f$. The kernels $\eta(s)$ and $\nu(s)$ are defined by :

$$\nu(s) = \int_0^\infty \frac{d\omega}{\pi} I(\omega) \coth(\frac{\hbar\omega}{2kT}) \cos\omega s \qquad (2.4)$$

$$\eta(s) = \frac{d}{ds}\gamma(s) \qquad (2.5)$$

where

$$\gamma(s) = \int_0^\infty \frac{d\omega}{\pi} \frac{I(\omega)}{\omega} \cos\omega s \qquad (2.6)$$

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and $I(\omega)$ is the spectral density,

$$I(\omega) = \sum_n \delta(\omega - \omega_n) \frac{\pi C_n^2}{2m_n\omega_n} \tag{2.7}$$

The kernel $\nu(s)$ modifies the action of the distinguished system and leads to dissipation and renormalization of the potential. The kernel $\eta(s)$ is responsible for noise and the process of decoherence. Both kernels are completely determined once a form for the spectral density is specified. A choice that proves to be convenient is

$$I(\omega) = M\gamma\omega(\frac{\omega}{\Lambda})^{s-1}exp(-\frac{\omega^2}{\Lambda^2}) \tag{2.8}$$

where Λ is a cut-off, which is usually taken to be very large.

The ohmic case ($s = 1$) is of particular interest. Taking Λ to infinity gives

$$\gamma(s) = M\gamma\delta(s)$$

which means that the corresponding classical equations of motion are local in time. The noise kernel is non-local for large Λ, except in the Fokker-Planck limit, $kT >> \hbar\Lambda$, in which we have,

$$\nu(s) = \frac{2M\gamma kT}{\hbar}\delta(s) \tag{2.9}$$

In this regime the reduced density matrix ρ (we drop the tilde) satisfies the master equation

$$\begin{aligned}
\frac{\partial\rho}{\partial t} = & -\frac{\hbar}{2Mi}(\frac{\partial^2\rho}{\partial x^2} - \frac{\partial^2\rho}{\partial y^2}) - \gamma(x-y)(\frac{\partial\rho}{\partial x} - \frac{\partial\rho}{\partial y}) \\
& + \frac{1}{i\hbar}[V(x) - V(y)]\rho - \frac{D}{\hbar^2}(x-y)^2\rho
\end{aligned} \tag{2.10}$$

where we set $D = 2M\gamma kT$, or in operator form :

$$\begin{aligned}
\frac{\partial\rho}{\partial t} & = \frac{1}{i\hbar}[\frac{p^2}{2M} + V(x), \rho] \\
& - \frac{\gamma}{i\hbar}[x, \{\rho, p\}] - \frac{D}{\hbar^2}[x, [x, \rho]] \\
& \equiv \mathcal{L}[\rho]
\end{aligned} \tag{2.11}$$

This master equation describes a Markovian process. The time evolution of the initial density matrix can be represented by the action of a one-parameter semigroup with generator $\mathcal{L}$ on the state space.

We will represent an element of the semigroup as $e^{\mathcal{L}t}$ and the time evolution of an initial density matrix ρ_0 as $e^{\mathcal{L}t}[\rho_0]$.

3 Phase space projectors

3.1 General considerations

In order to discuss phase space histories, we have to find elements of the quantum mechanical formalism that correspond closely to the notion of a phase space cell, at least in the macroscopic domain. Since any proposition about a physical system at a given instant of time can be represented as a projection operator, what we need is a class of operators that project onto the subspace of physical states which have position and momentum well within some phase space cell Γ.

Position and momentum cannot be simultaneously specified with arbitrary accuracy, so we must consider approximate projection operators . We call an operator P a quasiprojector into a phase space cell Γ if any wave function well localized inside Γ (outside Γ) is an approximate eigenfunction of P with eigenvalue 1 (0) . The construction of such operators is particularly easy in the Weyl representation, in which we associate with P a function $f(x, \xi)$ on a mock phase space, which we call its associated symbol

$$f(x,\xi) = \int dy \exp(-i\xi y/\hbar)\langle x + \frac{y}{2}|P|x - \frac{y}{2}\rangle \tag{3.1}$$

In general, a function $f(x, \xi)$ on a two dimensional phase space is called a symbol of order m, where m is real, if its derivatives of all orders have the following bounds

$$|\partial_x^\alpha \partial_\xi^\beta f(x,\xi)| \le C_{\alpha\beta}(1 + x^2 + \xi^2)^{(m-\alpha-\beta)/2} \tag{3.2}$$

where $C_{\alpha\beta}$ are constants called the seminorms of f . Note also, that x and ξ are in dimensionless units (scaled by some typical dimensions L and P).

One possibility is to take for the symbol corresponding to a quasi-projector on a phase space cell Γ the characteristic function of the cell Γ. This does not define a good operator because of the discontinuity at the boundary so instead, we consider a smoothened characteristic function [4]. That is, we take f to be equal to 1 inside and to 0 outside Γ, except in a small region along the boundary of Γ, where it interpolates smoothly from 1 to 0. This transition region is called the margin M of the cell Γ . In order for the operator to be well defined, the phase space cell has to be regular; loosely speaking it should not develope structure on the scale of $\hbar$ and the volume of the margin M

should be much smaller than the volume of Γ. The parameter $\epsilon = [M]/[\Gamma]$ is a measure of the validity of the approximation. For a sufficiently regular cell it is of the order of $(\hbar/LP)^{\frac{1}{2}}$, where L, P are typical length and momentum scales for our cell. We will return to the

issue of regularity in more detail later. The quasiprojectors P have a number of interesting properties [3]:

1. Their trace is proportional to the volume of the phase-space cell.

$$TrP = [\Gamma]/(2\pi\hbar) \tag{3.3}$$

2. They are very close to true projectors:

$$Tr|P - P^2| < c\epsilon \tag{3.4}$$

where c is a number of the order of unity.

3. For two different projectors on the same phase space cell P and P' with corresponding parameters ϵ and ϵ' we have :

$$Tr|P - P'| < c(\epsilon + \epsilon') \tag{3.5}$$

4. If Γ_1, Γ_2 and $\Gamma_1 \cap \Gamma_2$ are three phase space cells and P_1, P_2, P_{12} quasiprojectors associated to them, the operator $\delta P = P_1 P_2 - P_{12}$ is bounded in trace norm by :

$$Tr|\delta P| < K\frac{[\Gamma]}{2\pi\hbar}(\hbar/LP)^\alpha \tag{3.6}$$

where $[\Gamma] = sup([\Gamma_1], [\Gamma_2])$, $\alpha > \frac{1}{2}$ and K of order unity.

3.2 Coherent state quasiprojectors

One particular way to construct quasiprojectors is through the use of coherent states [3]. Consider the gaussian wave function :

$$g_{qp}(x) = (\frac{\Sigma}{2\pi\hbar})^{\frac{1}{4}} \exp(-\frac{\Sigma}{4\hbar}(1 + ir)(x - q)^2 + ipx/\hbar) \tag{3.7}$$

This induces a metric :

$$d(x, p) = \frac{\Sigma}{4\hbar}x^2 + \frac{\hbar}{\Sigma(1 + r^2)}p^2 \tag{3.8}$$

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on phase space with respect to which one can give a precise definition of a regular phase space cell [3, 4].

We call a phase space cell regular to order ϵ, with respect to one particular family of coherent states if:

1. The curvature radii with respect to the metric d of $\partial\Gamma$ are larger than l in absolute value.

2. The margin of the cell M which is defined as :

$$M = \bigcap_{(x,p)\epsilon\partial\Gamma} e(x, p, l)$$

where $e(x, p, l)$ is the ellipsis defined by :

$$e(x, p, l) = \{(x', p')|d(x - x', p - p') < l^2\}$$

satisfies $[M] < \epsilon[\Gamma]$.

3. The numbers ϵ and l satisfy : $e^{-2l^2} < \epsilon$.

If Γ is a regular cell to order ϵ then the operator :

$$P = \int_{\Gamma} \frac{dqdp}{2\pi\hbar} |g_{qp}\rangle \langle g_{qp}| \tag{3.9}$$

is a quasiprojector associated with Γ. We can readily see that its symbol :

$$f(x, \xi) = \int_{\Gamma} \frac{dqdp}{\pi\hbar} \exp[-\frac{\Sigma}{2\hbar}(1 + r^2)(x - q)^2$$
$$-\frac{2}{\hbar\Sigma}(\xi - p)^2 - \frac{2\Sigma r}{\hbar}(x - q)(\xi - p)] \tag{3.10}$$

is up to corrections of order e^{-l^2} equal to 1 (0) inside (respectively outside) Γ except for the margin M where it interpolates between those two values. It also satisfies the conditions for being a symbol of arbitrary negative order.

3.3 Generalized gaussian projectors

We are mainly interested in the time evolution of the quasiprojectors, when our system is under the influence of the thermal environment, as in the

case of quantum Brownian motion models. Due to the non-unitarity of the evolution, the operator in time t will take a form that cannot be related to the quasiprojectors (3.9) in a straightforward way, as in [3]. Omnès has shown that under a large class of potentials, the density operators of the form $|g_{qp}\rangle\langle g_{qp}|$ evolve approximately into operators of the same form but belonging to a different family of coherent states (different parameters Σ and r). The non - unitarity of the evolution in our case implies that an operator of this form become mixed. Therefore, we need a larger class of gaussian density operators.

We propose a class of quasiprojectors, that are defined through gaussian density matrices, which we choose to parametrize by the set of five numbers (Σ, F, r, q, p) as

$$\langle x|\rho|y\rangle = (\frac{\Sigma}{2\pi\hbar})^{\frac{1}{2}} \exp[-\frac{\Sigma}{2\hbar}(\frac{x+y}{2} - q)^2$$
$$-\frac{F}{2\hbar}(x-y)^2 - \frac{ir\Sigma}{2\hbar}(\frac{x+y}{2} - q)(x-y) + i\frac{p}{\hbar}(x-y)] \qquad (3.11)$$

which is defined for $\Sigma \leq 4F$ in order to satisfy the positivity requirement. For these density matrices we can verify that,

$$\|\rho\|_{H.S.} = (Tr\rho^2)^{\frac{1}{2}} = (\Sigma/4F)^{\frac{1}{2}} = (\hbar/2\mathcal{A}) \leq 1 \qquad (3.12)$$

where $\mathcal{A}$ is the Wigner function area defined as :

$$\mathcal{A}^2 = (\Delta q)^2 (\Delta p)^2 - C_{pq}^2$$

with :

$$C_{pq} = \frac{1}{2}\langle (q - \langle q \rangle)(p - \langle p \rangle) + (p - \langle p \rangle)(q - \langle q \rangle)\rangle$$

$\mathcal{A}$ is a measure of the phase space area in which the gaussian density matrix is localized.

This density matrix defines a metric on phase space in analogy to (3.8):

$$d(x,p) = \frac{\Sigma}{4\hbar}x^2 + \frac{\hbar}{4F(1 + \frac{\Sigma}{4F}r^2)}p^2 \qquad (3.13)$$

With respect to this metric, we can define the notion of the margin of a phase space cell and the notion of regularity (up to order ϵ) as before . We

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define the operator :

$$P = \int_\Gamma \frac{dqdp}{2\pi\hbar} \quad \rho(\Sigma, F, r, q, p) \qquad (3.14)$$

This is a quasiprojector associated with the cell Γ.

To see this, consider its symbol,

$$f(x, \xi) = (\Sigma/4F)^{\frac{1}{2}} \int_\Gamma \frac{dqdp}{\pi\hbar} \exp[-\frac{\Sigma}{2\hbar}(1 + \frac{\Sigma}{4F}r^2)(x - q)^2$$
$$-\frac{1}{2\hbar F}(\xi - p)^2 - \frac{\Sigma r}{2\hbar F}(x - q)(\xi - p)] \qquad (3.15)$$

It is a symbol of arbitrary negative order and has the form of the most general gaussian smeared characteristic function associated to the cell Γ. We can verify that it takes values 1 (0) inside (outside) Γ up to corrections of order e^{-l^2}, where l is defined as before but with respect to the metric (3.13). It interpolates between those two values only in the margin, which has a width of order $\mathcal{A}^{\frac{1}{2}}$.

How can we compare this class of projectors to the ones defined through the general gaussian states (3.11)? A coherent state is localized within a volume of order $\hbar/2$ and is unable to distinguish points found within a volume of this size. For this reason we expect the margin associated with a projector onto a phase space cell to have an area of order $\hbar$ and the parameter ϵ to be of order $(\hbar/LP)^{\frac{1}{2}}$. This estimation is backed by a detailed calculation using microlocal analysis, that is, we can verify that this order of magnitude for the parameter ϵ is the one that gives the best approximation of a quasiprojector to an exact projector [3, 4].

On the other hand the gaussian density matrix (3.11) is localized within a volume of order $\mathcal{A}$. Therefore we expect the width of the margin to be of order $\mathcal{A}$ and the parameter ϵ to be of order $(\mathcal{A}/LP)^{\frac{1}{2}}$ for the class of projectors (3.14). This is clearly, not the "best" quasiprojector one can associate to a phase space cell, but this generalization is necessary. As long as the quantity $(2\mathcal{A}/\hbar)^{\frac{1}{2}}$ is of the order of unity we still have a good approximation for a phase space projector. The larger the value of $\mathcal{A}$ is, the worse the degree of approximation to a true projector is. We are therefore going to refer to projectors with minimum value of $\mathcal{A} = \hbar/2$ (the ones constructed from pure gaussians) as the maximum resolution projectors.

We will see that the time evolution of an initial generalized coherent state remains within a good approximation a gaussian density matrix, with time increasing value of $\mathcal{A}$. We know from the study of linear systems that the value of $\mathcal{A}$ increases polynomially in the short time limit, to become constant for times much larger than the relaxation time γ^{-1} [12]. Its asymptotic value corresponds to the uncertainty due to thermal fluctuations, and is typically many orders of magnitude larger than the quantum ones. Still, for sufficiently large phase space cells $\mathcal{A}$ is much smaller than the size of the cell, the quantity ϵ remains quite small and our operators (3.14) are still close to true projectors. On the other hand, for smaller cells for which $\hbar << LP < \mathcal{A}_\infty$ there comes some time (usually of the order of γ^{-1}), when the evolved quasiprojectors cannot anymore distinguish the phase space cell, since the size of the margin has become essentially as large as the whole of the cell.

4   Time evolution of gaussian density matrices

We are interested in the evolution of the gaussian density matrices (3.11) in the class of quantum Brownian motion models described by the master equation (2.11). As we mentioned, this expression is valid in the Markovian regime (high temperature and ohmic environment).

For linear potentials, we know (see for instance [9, 12]) that the propagation is gaussian, and an initial gaussian density matrix remains gaussian centered around the classical path. Also in the case of zero coupling to the environment (unitary evolution) Hagedorn [13] has established that a generalized coherent state retains its shape for a period of time and its center follows the classical equations of motion. The error of this approximation is of the order of $(\hbar/LP)^\lambda$, where $\lambda < \frac{1}{2}$.

We seek a generalization of these results. We want to show, that for a large class of potentials $V(x)$ the master equation (2.11) respects the gaussian nature of the density matrices (3.11) and that their centers follow the classical equations of motion (with dissipation). We shall show that this is a good approximation as long as the spread in position is sufficiently small and that the error is of the order of $(\hbar/LP)^\lambda$ as in the purely hamiltonian case.

4.1 The theorem

In this section, we present our result in the form of a theorem, a detailed proof of which can be found in the Appendix. We assume that the potential $V(x)$ satisfies:

1. $V^{(2)}(x)$ is continuous and uniformly Lipschitz on compact subsets of $\mathcal{R}$ (i.e. given any $R > 0$ there exists k such that $|V^{(2)}(x) - V^{(2)}(y)| < \beta|x - y|$ whenever $|x| < R$, $|y| < R$.
2. $|V(x)| < e^{Mx^2}$
3. $V(x)$ is bounded from below.

Assume that at $t = 0$ the system is in the state $\rho_0 = \rho(\Sigma_0, F_0, r_0, q_0, p_0)$. The exact solution to the master equation (2.11) can be writen formally as :

$$\rho_t = e^{\mathcal{L}t}[\rho_0]$$

where $\mathcal{L}$ is the generator of the one-parameter semigroup acting on the state space, which is determined by the dynamics.

Then we have the following theorem :

Theorem: For each $T > 0$, $N > 0$, $p < \frac{1}{2}$, $\alpha < \frac{1}{2} - p$ and $0 < t < T$, such that

$$\left(\frac{\Sigma F}{\Sigma + 4F}\right)^{-1} < N\hbar^{-2p}$$

there exists $C > 0$, $\delta > 0$ for which :

$$\|e^{\mathcal{L}t}[\rho_0] - \rho(\Sigma(t), F(t), r(t), q(t), p(t))\|_{HS} < C\hbar^\lambda \qquad (4.1)$$

whenever $\hbar < \delta$. We denote $\lambda = 3\alpha - 1$.

The quantities $(\Sigma(t), F(t), r(t), q(t), p(t))$ are solutions to the system of differential equations :

$$\dot{q} = \frac{p}{M} \qquad (4.2)$$

$$\dot{p} = -V'(q) - 2\gamma p \qquad (4.3)$$

$$\dot{\Sigma} = \frac{1}{M}\Sigma^2 r \qquad (4.4)$$

$$\dot{F} = \frac{1}{M}\Sigma F r - 4\gamma F + \frac{2D}{\hbar} \qquad (4.5)$$

$$\dot{r} = -\frac{\Sigma r^2}{2M} - \frac{2}{M}F - 2\gamma r + \frac{2}{\Sigma}V^{(2)}(q) \qquad (4.6)$$

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under the initial conditions :

$$(\Sigma(0), F(0), r(0), q(0), p(0)) = (\Sigma_0, F_0, r_0, q_0, p_0) \tag{4.7}$$

Note that we have used dimensionless units with $L = 1$ and $P = 1$.

The proof is based on the observation that the generator $\mathcal{L}$ can be split into two parts, described in the Appendix: $\mathcal{L} = \mathcal{S} + \mathcal{U}$, where $\mathcal{S}$ is the generator that corresponds to the Brownian motion of a free particle and $\mathcal{U}$ a generator that contains only the effects of the potential. The action of $\mathcal{S}$ can be exactly determined (for instance using the density matrix propagator which can be exactly computed for the free particle), while the action of $\mathcal{U}$ can be approximated with a quadratic expression. Thus we get expressions for $e^{\mathcal{S}t}[\rho_0]$ and $e^{\mathcal{U}t}[\rho_0]$ which can be combined to give an expression for $e^{\mathcal{L}t}[\rho_0]$ through the use of the Trotter product formula. The proof follows closely the one of Hagedorn [13], with the main differences being the non-unitarity of the generator $\mathcal{S}$ and that our Hilbert space is the one of the density matrix under the Hilbert-Schmidt norm.

The proof involves taking a Taylor expansion of the potential and keeping only the quadratic terms. Therefore our results are exact for the harmonic oscillator and the free particle case.

4.2 Uncertainties

We can write the uncertainties in position Δq and in momentum Δp as well as the correlation C_{pq} associated with the gaussian density matrix in terms of the parameters Σ, r and F:

$$(\Delta q)^2 = \hbar/\Sigma \tag{4.8}$$

$$(\Delta p)^2 = \hbar F(1 + \frac{\Sigma}{4F}r^2) \tag{4.9}$$

$$C_{pq} = -\frac{\hbar r}{2} \tag{4.10}$$

From this we can see the interpretation of the parameters Σ, F and r we used to parametrize the gaussian density matrices. Clearly Σ^{-1} is a measure of the spread in position and r of the correlation between position and momentum. Σ/F is proportional to $Tr\rho^2$ and therefore is a measure of the non-purity of the state. The original parameters Σ, F, r have been more convenient in

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the course of the proof, but in order to have a clearer interpretation of the results, we shift our attention to the set of variables Δq, Δp, C_{pq}. Within our approximation they evolve according to the equations:

$$\frac{d}{dt}(\Delta q)^2 = \frac{2}{M}C_{pq} \tag{4.11}$$

$$\frac{d}{dt}(\Delta p)^2 = -4\gamma(\Delta p)^2 - 2C_{pq}V^{(2)}(q) + 2D \tag{4.12}$$

$$\frac{d}{dt}C_{pq} = \frac{1}{M}(\Delta p)^2 - 2\gamma C_{pq} - (\Delta q)^2 V^{(2)}(q) \tag{4.13}$$

We notice, that the diffusion coefficient appears only in the equation

for the momentum uncertainty, and at short times is the dominant term. This means that the spread in momentum is more effective than the corresponding spread in position, which depends on the diffusion coefficient only indirectly. It is also important to stress the reluctance of particles with large mass to undergo a rapid growth of the fluctuations.

4.3   Validity of the approximation

The essential requirement for the validity of our approximation is that the quantity $\frac{\Sigma+4F}{\Sigma F}$ remains sufficiently large through the evolution according to the equations (4.2-4.6). Since we have that $\Sigma \leq 4F$, this is equivalent to saying that Σ remains sufficiently large (much larger than $N\hbar^{2p}$). But a large value of Σ means that the particle is well localized in position. We can thus say, that while the state remains well localized in position (that means $\Delta q < N^{-\frac{1}{2}}\hbar^{\frac{1}{2}-p}$), the gaussian approximation will be quite good. When, mainly due to the diffusion, the state of the particle has become spread in space, the large scale structure of the potential becomes important and even weak non-linearities will contribute significantly in the evolution, thus rendering the gaussian approximation invalid. Note, that the constants λ and p satisfy :

$$\lambda < \frac{1}{2} - 3p$$

which shows that the larger our tolerance for the spread in position, the larger the error stemming from our approximation.

In general, we cannot say much about the time when the gaussian approximation breaks down, since this depends crucially on the form of the potential, the particle mass and the position and spread of the initial gaussian.

Excluding cases of potentials varying significantly in the microscopic scale or potentials where tunnelling effects can be important, we expect the gaussian approximation to hold within a very good accuracy for times much smaller than the typical time scale in our dynamics : γ^{-1}. For larger times, the validity of the approximation, depends heavily on the length scale L on which $V^{(2)}(x)$ varies. If this length scale is much larger than the size of the thermal fluctuations the approximation will hold for timescales some orders of magnitude larger than γ^{-1}. If this is not the case, the approximation will have broken down much earlier. Within a few times γ^{-1} the particle will tend towards the state of thermal equilibrium.

We can clarify those ideas by examining the simplest case of a system moving in a potential exhibiting only weak nonlinearities :

$$V(x) = \frac{1}{2}M\omega^2 x^2 + \eta x^4 \tag{4.14}$$

This effect of the non-linearities becomes significant at length scales of the order of

$$L = (\frac{M\omega^2}{\eta})^{\frac{1}{2}}$$

Assuming weak coupling to the environment, the uncertainty in position is given in leading order to ϵ and γ/ω [12] :

$$(\Delta q)^2 = (\Delta q')^2 e^{-2\gamma t} + \frac{MkT}{\omega^2}(1 - e^{-2\gamma t}) \tag{4.15}$$

with $(\Delta q')^2$ containing the effects of the unitary evolution. The gaussian approximation will break down when $\Delta q \simeq L$. We can readily verify that for $\eta << \omega^4/kT$, Δq will remain much smaller than L, while for larger values of η the approximation will break down at a time scale of the order of γ^{-1}.

We should note, that for a large variety of physical systems, it is not necessary to assume weak nonlinearities, in order to have a large value of L. Systems with potentials corresponding to a spatial average of many microscopic degrees of freedom have typical values for L that can be said to correspond to a macroscopic scale.

A relevant question is to give an estimation of the error term $C\hbar^\lambda$ of our approximation. For the case of the closed system, a very rough estimation is

$$\hbar^\lambda \int_0^t dt |V^{(3)}(q(t))| \Delta\dot{q}(t)$$

in dimensionless units [3]. When considering the open system case we can see from the detailed study of the proof, that a term of $Tr\rho^2 < 1$ enters the right hand side of the inequalities, thus rendering our approximation better. Also for U-shaped potential the effect of the dissipation is to make the values of the third derivatives of the potential smaller in absolute value. For the time, that the increase in the uncertainty due to the diffusion is small, we expect our approximation to be better when taking into account the coupling to the environment. On the other hand, at times larger than the typical time where thermal fluctuations overcome the quantum ones, the more effective spread of the density matrix makes the effect of non-linearities more important, and thus reduces the accuracy of the gaussian approximation. Therefore, we arrive at the following picture:

At times less than $(\hbar/\gamma kT)^{\frac{1}{2}}$ the gaussian approximation is better when the environment is taken into account.

At larger times because of the effects of diffusion, the approximation becomes gradually worse, since diffusion begins to affect strongly the uncertainty in the position.

Finally we should make a remark on a problem that may arise when having a closer look at the proof of the theorem. We know that the master equation (2.11) does not preserve positivity at very short time scales [12, 14]. This is a result of taking the cut-off Λ to infinity. Our proof, being based on a discretization of the time and taking the continuous limit in the end, might be inadequate for a class of initial states. We can avoid this problem, by adding a term $\eta[p,[p,\rho]]$ in the master equation (like the one appearing in [15]). The master equation thus becomes positive and this term does not change the nature of the proof. In the end we can set η equal to zero, an approximation that is valid in the high temperature regime.

5   Evolution of quasiprojectors

In this section we study the time evolution of the quasiprojectors P introduced in Section 3. We shall show that as long as the corresponding phase space cells remain large and regular, they evolve according to the classical equations of motion with dissipation. One expects that the coupling to an environment will induce a degradation in the quasiprojectors as they evolve. This is reflected in an increase of the size of the margin, which implies a loss of predictability.

We work in the equivalent of the Heisenberg picture for open systems. That is, we assume that the density matrix is time independent and that the effect of the evolution is contained in the operators. Let us further assume that in this picture a (bounded) operator P evolves under the action of an one parameter semigroup with generator $\mathcal{M}$. The simple correspondence between $\mathcal{M}$ and $\mathcal{L}$ that exists in unitary evolution is lost here.

We determine $\mathcal{M}$ by demanding that the probabilities are the same in both pictures. This means :

$$Tr(e^{\mathcal{M}t}[P]\rho) = Tr(Pe^{\mathcal{L}t}[\rho]) \tag{5.1}$$

which is equivalent to demanding :

$$Tr(\mathcal{M}[P]\rho) = Tr(P\mathcal{L}[\rho]) \tag{5.2}$$

Having the expression (2.11) for $\mathcal{L}$ it is easy to show that :

$$\begin{aligned}
\frac{\partial P}{\partial t} = \mathcal{M}[P] = \quad &- \quad \frac{1}{i\hbar}[\frac{p^2}{2M} + V(x), P] \\
&- \quad \frac{\gamma}{i\hbar}\{p, [P, x]\} - \frac{D}{\hbar^2}[x, [x, P]]
\end{aligned} \tag{5.3}$$

The relation of this equation to the one for ρ is more clarifying in the position representation :

$$\frac{\partial}{\partial t}P(x,y) = \frac{\hbar}{2Mi}(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2})P(x,y) + \gamma(x-y)(\frac{\partial}{\partial x} - \frac{\partial}{\partial y})P(x,y)$$

$$-\frac{1}{i\hbar}[V(x) - V(y)]P(x,y) - 2\gamma P(x,y) - \frac{D}{\hbar^2}(x-y)^2 P(x,y) \quad (5.4)$$

Comparing this equation to the one for the ρ, we notice the change of sign in the terms corresponding to the dynamical evolution and the appearance of the extra term $-2\gamma P$. Because of this term the equation fails to preserve the trace. This is an expected feature, when studying the evolution of quasiprojectors, since classical dissipative evolution does not preserve the phase space area.

We can study now the evolution of gaussian operators $W(\Sigma, F, r, q, p)$:

$$< x|W|y >= (\frac{\Sigma}{2\pi\hbar})^{\frac{1}{2}} \quad \exp[-\frac{\Sigma}{2\hbar}(\frac{x+y}{2} - q)^2$$
$$-\frac{F}{2\hbar}(x-y)^2 - \frac{ir\Sigma}{2\hbar}(\frac{x+y}{2} - q)(x-y) + \frac{ip}{\hbar}(x-y)] \qquad (5.5)$$

It is straightforward to show, using the treatment found in the appendix, that within an error of $C(\hbar/LP)^\lambda$ in the Hilbert-Schmidt norm,

$$e^{\mathcal{M}t}[W(\Sigma_0, F_0, r_0, q_0, p_0)] = J(t) \ W(\Sigma(t), F(t), r(t), q(t), p(t)) \qquad (5.6)$$

where $\Sigma(t)$, $r(t)$, $F(t)$, $q(t)$, $p(t)$ are the solutions to the equations,

$$\dot{q} = -\frac{p}{M} \qquad (5.7)$$

$$\dot{p} = 2\gamma p + V'(q) \qquad (5.8)$$

$$\dot{\Sigma} = -\frac{1}{M}\Sigma^2 r \qquad (5.9)$$

$$\dot{F} = -\frac{1}{M}\Sigma F r + 4\gamma F + \frac{2D}{\hbar} \qquad (5.10)$$

$$\dot{r} = \frac{\Sigma r^2}{2M} + \frac{2}{M}F + 2\gamma r - \frac{2}{\Sigma}V^{(2)}(q) \qquad (5.11)$$

under the conditions

$$(\Sigma(0), F(0), r(0), q(0), p(0)) = (\Sigma_0, F_0, r_0, q_0, p_0) \qquad (5.12)$$

The equations for q and p are the backwards classical equations of
motion (not the time reverse) and $J(t)$ is the Jacobian of the transformation from (q_0, p_0) to $(q(t), p(t))$

$$J(t) = \frac{\partial(q(t), p(t))}{\partial(q_0, p_0)} = e^{2\gamma t} \qquad (5.13)$$

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Now let us consider the evolution of a quasiprojector P_Γ, associated with a phase space cell Γ which is regular to order ϵ. Consider an operator that at $t = 0$ has maximum resolution in phase space. We thus have at $t = 0$

$$P_\Gamma = \int_\Gamma \frac{dp_0 dq_0}{2\pi\hbar} \quad W(\Sigma_0, \Sigma_0/4, r_0, q_0, p_0) \tag{5.14}$$

As it evolves under the action of $e^{\mathcal{M}t}$ this projector becomes

$$e^{\mathcal{M}t}[P_\Gamma] = J(t) \int_\Gamma \frac{dp_0 dq_0}{2\pi\hbar} \quad W(\Sigma(t), F(t), r(t), q_{cl}(-t), p_{cl}(-t)) \tag{5.15}$$

within an error of order

$$(\hbar/LP)^\lambda Tr P_\Gamma$$

Performing a transformation from (q_0, p_0) to $(q_{cl}(-t), p_{cl}(-t))$ we readily verify that :

$$e^{\mathcal{M}t}[P_\Gamma] = \int_{\Gamma_t} \frac{dq dp}{2\pi\hbar} W(\Sigma(t), F(t), r(t), q, p) = P'_{\Gamma_t} + O(\epsilon') \tag{5.16}$$

where Γ_t is the phase space cell obtained from Γ through the classical equations of motion and $\epsilon' = (\mathcal{A}/LP)^{\frac{1}{2}}$ determines the degree of regularity of Γ_t. Here we have:

$$\mathcal{A} = \frac{\hbar}{2} \frac{4F(t)}{\Sigma(t)} \tag{5.17}$$

is the Wigner function area. P' is clearly another quasiprojector associated with the phase space cell Γ_t. Essentially we have :

$$\|e^{\mathcal{M}t}[P_\Gamma] - P'_{\Gamma_t}\|_{HS} < c(\hbar/LP)^{\frac{1}{2}} Tr P_\Gamma \tag{5.18}$$

From (3.4) we know that within an error of ϵ', P'_Γ is equal to P_{Γ_t}, where P_{Γ_t} is the quasiprojector defined through the gaussian operator with parameters $(\Sigma_0, 4\Sigma_0, r_0)$

$$\|P'_{\Gamma_t} - P_{\Gamma_t}\|_{HS} \leq Tr|P'_{\Gamma_t} - P_{\Gamma_t}| < c'\epsilon' Tr P_{\Gamma_t} \tag{5.19}$$

Assuming that during the evolution the error of the gaussian approximation is less than ϵ', we conclude from (5.18) and (5.19) that

$$\|e^{\mathcal{M}t}[P_\Gamma] - P_{\Gamma_t}\|_{HS} < C\epsilon' Tr P_\Gamma \tag{5.20}$$

This is our main result: A projector P_Γ onto some phase space cell Γ evolves under the action of the dynamical semigroup into a projector P_{Γ_t} associated with the phase space cell Γ_t, which is obtained from Γ through the classical dissipative equations of motion. This is an approximate result, accurate within an error of order ϵ'. This parameter increases with time solely because of the noise induced by the environment. This implies that the margin of the quasiprojector increases with time evolution and the predictability becomes gradually worse.

We can say that the effective region in phase space occupied by a projector P_Γ consists of the cell Γ , plus the margin. Thus the effective region the projector occupies increases in a way that has nothing to do with the corresponding classical equations of motion (unlike the shrinking due to dissipation which is a classical feature). A measure for this increase, which can be attributed solely to the diffusion, is

$$\mu = \frac{[\Gamma_t] + [M']}{[\Gamma_t] + [M]} = \frac{1 + \epsilon'}{1 + \epsilon} \simeq 1 + \epsilon' - \epsilon \simeq 1 + (\frac{\mathcal{A}}{LP})^{\frac{1}{2}}(1 - (\hbar/\mathcal{A})^{\frac{1}{2}}) \quad (5.21)$$

We can view this result as follows: Sufficiently coarse grained observables (projectors onto large phase space cells) evolve under classical deterministic equations plus the action of noise terms. These terms, stemming from the ignored degrees of freedom of both the system and the environment induce an error of order ϵ' to the classical equations of motions. The parameter measuring the size of the noise terms and correspondingly the loss of predictability is the Wigner function area $\mathcal{A}$ [12], which in general is an increasing function of time.

It may be shown that in the short time limit $(t << \gamma^{-1})$ for all initial operators (5.14) it is given by [12] :

$$\mathcal{A}^2(t) = \frac{\hbar^2}{4} + \frac{32}{3}\frac{\gamma^2 k^2 T^2}{\hbar^2}t^4 \tag{5.22}$$

We obtain the same result, when considering the backward time evolution of the quasiprojectors.

6   Histories

6.1 Construction of the decoherence functional

Using our previous results we now construct the phase space histories for the Brownian particle and study their decoherence properties.

We have found how the quasiprojectors associated with phase space cells evolve in time. We can use this knowledge in the calculation of the decoherence functional. Our results are of use only when it can be written as:

$$D(\alpha, \alpha') = Tr_s[P^n_{\alpha_n} K^{t_n}_{t_{n-1}}[\ldots P^1_{\alpha_1} K^{t_1}_{t_0}[\tilde{\rho}] P^1_{\alpha'_1} \ldots] P^n_{\alpha'_n}] \qquad (6.1)$$

in terms of a superoperator $K^{t'}_t$ in the space of the reduced density matrices. This is actually valid in the Markovian regime [16].

In the Caldeira -Leggett model the environment is taken to be essentially infinite and thus the environmental degrees of freedom remain very close to the state of thermal equilibrium. The correlations created will in general be of the order of Λ^{-1} and since the evolution has no memory their effect will be negligible in a time scale of order Λ^{-1}. Remembering that in our regime we have assumed $kT >> \hbar\Lambda$, we conclude that if the time interval between the two projections satisfies :

$$t_2 - t_1 > O(\hbar/kT)$$

we can write the decoherence functional for two-time histories as

$$D(\alpha, \alpha') = \delta_{\alpha_2 \alpha'_2} Tr(P^2_{\alpha_2} e^{\mathcal{L}(t_2 - t_1)}[P^1_{\alpha_1}] \tilde{\rho}_{t_1} P^1_{\alpha'_1}) \qquad (6.2)$$

or using the definition of the generator $\mathcal{M}$

$$D(\alpha, \alpha') = \delta_{\alpha_2 \alpha'_2} Tr(e^{-\mathcal{M}(t_2 - t_1)}[P^2_{\alpha_2}] P^1_{\alpha_1} \tilde{\rho}_{t_1} P^1_{\alpha'_1}) \qquad (6.3)$$

We can assign a probability measure

$$p(C_\alpha) = Tr((e^{-\mathcal{M}(t_2 - t_1)}[P^2_{\alpha_2}] P^1_{\alpha_1} \tilde{\rho}_{t_1} P^1_{\alpha_1}) \qquad (6.4)$$

to the history C_α if the probability sum rules are satisfied. A sufficient condition for this is the vanishing of the off-diagonal elements of the decoherence functional.

6.2 Approximate decoherence and determinism

The sum rules are not satisfied exactly in our case, but only within an error of order ϵ. A good condition for this approximate consistency is :

$$|Tr(e^{-\mathcal{M}(t_2-t_1)}[P_{\alpha_2}^2]P_{\alpha_1}^1\tilde{\rho}_{t_1}P_{\alpha_1'}^1)| <$$
$$\epsilon \quad \sum_{\alpha_1} Tr(e^{-\mathcal{M}(t_2-t_1)}[P_{\alpha_2}^2]P_{\alpha_1}^1\tilde{\rho}_{t_1}P_{\alpha_1}^1) \tag{6.5}$$

Let us concentrate on the particular case of histories that correspond to the evolution of phase space cells along the classical trajectories. That means, we consider the set of histories $C_1 = (P_1, P_2)$ and $C_2 = (\bar{P}_1, P_2)$ where P_1 and P_2 are quasiprojectors associated with the cells Γ_1 and Γ_2 and Γ_2 is the cell obtained from the classical evolution of Γ_1. In order to achieve minimum error in our estimations we restrict our quasiprojectors to have minimum resolution in phase space. The condition (6.8) reads :

$$Tr(e^{-\mathcal{M}(t_2-t_1)}[P_2]P_1\tilde{\rho}_{t_1}\bar{P}_1) <$$
$$\epsilon(Tr((e^{-\mathcal{M}(t_2-t_1)}[P_2]P_1\tilde{\rho}_{t_1}P_1) + Tr(e^{-\mathcal{M}(t_2-t_1)}[P_2]\bar{P}_1\tilde{\rho}_{t_1}\bar{P}_1)) \tag{6.6}$$

From the analysis in the previous section, we know that within an error of order $(\mathcal{A}(t_2 - t_1)/LP)^{\frac{1}{2}}TrP_1$

$$e^{-\mathcal{M}(t_2-t_1)}[P_2] = P_1 \tag{6.7}$$

This means that the left hand side in (6.6) is of order

$$(\mathcal{A}(t_2 - t_1)/LP)^{\frac{1}{2}}TrP_1$$

and the right hand side of order

$$\epsilon Tr(P_1\tilde{\rho}_{t_1}) < \epsilon TrP_1$$

Thus the probability sum rules are satisfied within an order

$$\epsilon = (\mathcal{A}(t_2 - t_1)/LP)^{\frac{1}{2}}$$

$\mathcal{A}$ takes values between $\hbar/2$ and its asymptotic value which is of the order of the thermal fluctuations in phase space. Therefore, for a macroscopic phase space cell ϵ is a small number and decoherence is good. It is clear that the

degree of decoherence is better in the case of a hamiltonian system. We can thus compute the conditional probability, that the particle being within Γ_1 at time t_1 will be within Γ_2 at time t_2. This is clearly,

$$p(\Gamma_1, t_1 \rightarrow \Gamma_2, t_2) = 1 - O(\epsilon) \tag{6.8}$$

Therefore up to an order of ϵ there is good agreement between the predictions of classical and quantum dynamics for the evolution of the Brownian particle. This agreement is best when the phase space cells are regular and the time evolution preserves this regularity. These results are independent of the initial state of the system. One might expect that the degree of decoherence would be in inverse relation to the degree of predictability. Here, they are essentially the same, as in the case where the environment is not present.

The extension of this result to n-time histories is straightforward. Consider the history $C = (P_1, t_1; P_2, t_2; \ldots; P_n, t_n)$ where P_i is the maximum

resolution quasiprojector associated with a cell Γ_i and Γ_{i+1} is obtained from Γ_i

through the classical equations of motion. We construct from C a set of histories by replacing one or

more of the P_i 's with $\bar{P}_i$, and define the parameter $\epsilon = \max(\mathcal{A}(t_i - t_{i-1})/LP)^{\frac{1}{2}}$. If we assume that for the intermeddiate times

$$\min(t_{i+1} - t_i) > O(\hbar/kT)$$

and that the dynamics preserve the regularity of the cells to an order ϵ, then this set of histories satisfies the probability summation rules to this order.

7   Conclusions

In this paper, we have studied histories corresponding to the evolution o f phase space cells in quantum Brownian motion models with ohmic dissipation and Markovian dynamics. We constructed these histories using the generalization (3.14) of the quasiprojectors used by Omnès. The estimation (4.1) for the validity of the gaussian approximation has enabled us to establish, how those quasiprojectors evolve.

We showed that as long as the phase space cell is large and regular, histories corresponding to the evolution according to the classical deterministic

equations of motion approximately satisfy the sum rules. The order of magnitude of the corresponding error depends only on the Wigner function area $\mathcal{A}$ (a function only of the time difference between projections). $\mathcal{A}(t)$ can be computed as the solution of the system (5.7 - 5.11) under the initial condition (5.12). It starts as containing purely quantum uncertainties

at $t = 0$ and at a timescale of γ^{-1} becomes of the order of magnitude of its asymptotic value, which corresponds to thermal fluctuations.

There are three timescales in our model, which determine qualitative changes in the description of the dynamics: the Markovian time $t_M = \hbar/kT$, the decoherence time $t_d = (\hbar/\gamma kT)^{\frac{1}{2}}$ and the relaxation time $t_r = \gamma^{-1}$. For times $t << t_M$ dynamics are not Markovian and therefore we cannot talk about predictability. For times t such that $t_M < t < t_d$, sufficiently large phase space cells evolve almost under deterministic equations of motion and the details of the potential are not of importance in the increase

of the fluctuations. At $t = t_d$ the thermal fluctuations overcome the quantum ones [12]. For $t > t_d$ the degree of decoherence and predictability gradually worsens and is destroyed for cells with area

less than the thermal uncertainty. For these cells any sense of predictability has been lost at $t = t_r$. In the contrary sufficiently macroscopic cells, continue to evolve within a good approximation according to the classical equations of motion even at times larger than γ^{-1}. Eventually at long times, approximately deterministic evolution will break down, because the dissipative nature of the evolution tends to shrink the phase space cells, until their area is of the order of magnitude of the thermal fluctuations. We therefore see that the breakdown of predictability is dependent mainly on the parameters of the environment and the structure of the potential is largely irrelevant, unlike the unitary case. This comes probably from the fact that we have considered an one dimensional problem. It is to be expected that in a system with more degrees of freedom, time parameters of the potential (i.e the Lyapunoff exponents of the classical solutions) will play a more important role in the determination of when classical predictability breaks down.

Perhaps contrary to our expectations, the coupling to the environment tends to make the decoherence properties of histories worse. This means that histories corresponding to evolution of phase space cells are not the ones that give the sharpest correspondence with a deterministic classical description. This is to be expected, since the classical description of quantum brownian

motion is that of a stochastic process in phase space.

Essentially, there are two mechanisms that produce decoherence. One is the interaction with an environment and the other is the existence of phase space projection operators that have an intrinsic almost deterministic time evolution. In the case of purely hamiltonian dynamics only the latter appears. In our case, we have both, but it is clear that again the latter is the dominant one. This is attributed to the simplicity of the corresponding classical equations, in which the effect of the environment is contained only within the term $-2\gamma p$. It is therefore natural that observables close to the ones used in purely Hamiltonian systems will decohere giving rise to approximate predictability.

With those results we have achieved three things: First, we have generalized the results of Omnès on classical determinism for a class of open systems. In fact, we can generalize our results are valid for any Markovian equation of the Lindblad type [18], with environment operators L_n linear in position and momentum. Second, we have verified the validity of classical equations of motion with linear dissipation, as a consequance of the underlying quantum mechanical evolution and sufficient coarse-graining. Third, taking quantum Brownian motion as a toy model, we have obtained a picture of the behaviour of collective variables in many body systems and the quasiclassical behaviour of these variables.

Finally, we would like to compare our results with the ones obtained in [17] using the quantum state diffusion picture for the quantum Brownian motion. For the case of two-time phase space histories $C = (\Gamma_1, \Gamma_2)$ and under the assumption that they decohere, the authors gave for the associated probability the expression :

$$p(C) = \int_{\Gamma_2} dp_2 dq_2 \int_{\Gamma_1} dp_1 dq_1 J(p_2, q_2, t_2 | p_1, q_1, t_1) f(p_1, q_1, t_1) \qquad (7.1)$$

where $f(p, q, t)$ is a classical probability distribution satisfying the Fokker-Planck equation and J is the associated propagator. We can verify that this result is equivalent to ours in the particular case of an harmonic potential, where the propagator is gaussian centered around the classical path. If the volume of the phase space cells is much larger than the Wigner function area $\mathcal{A}(t)$ associated

with the propagator [12], then the propagation is within an error of order $\epsilon = (\mathcal{A}/LP)^{\frac{1}{2}}$ a δ-function around the classical path and the conditional

probability is found

$$p(\Gamma_1, t_1 \to \Gamma_2, t_2) = 1 - O(\epsilon)$$

8 Aknowledgements

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A   Proof of the theorem

We are going to give the proof for the theorem stated in section 4. It follows the reasoning of the similar proof of Hagedorn for the case
of closed system, where the reader can refer for further details.

The evolution equation for the density matrices in the Markowian regime (2.11) corresponds to the action of an one-parameter semigroup with generator $\mathcal{L}$ on the state space . We can write :

$$\mathcal{L} = \mathcal{S} + \mathcal{U}$$

where $\mathcal{S}$ and $\mathcal{U}$ are generators of one parameter semigroups given by :

$$\mathcal{S}[\rho] = \frac{1}{i\hbar}[p^2/2M, \rho] + \frac{\gamma}{i\hbar}[x, \{\rho, p\}] - \frac{D}{\hbar^2}[x, [x, \rho]] \qquad (A.1)$$

$$\mathcal{U}[\rho] = \frac{1}{i\hbar}[V(x), \rho] \qquad (A.2)$$

Splitting $\mathcal{L}$ into $\mathcal{S}$ and $\mathcal{U}$ will enable us to use the Trotter product formula to determine $e^{\mathcal{L}t}$ in terms of $e^{\mathcal{S}t}$ and $e^{\mathcal{U}t}$.

We want to establish, how the class of states described by the gaussian density matrices (3.11) evolve under $\mathcal{S}$ and $\mathcal{U}$.

1. Evolution under S

Let us consider the solution of :

$$\frac{\partial \rho}{\partial t} = \mathcal{S}[\rho]$$

with initial conditions

$$\rho = \rho(\Sigma_0, F_0, r_0, q_0, p_0)$$

at time $t = 0$. This is just the solution of the master equation for the free particle. For short times t ($t < \tau << \gamma^{-1}$) we have that :

$$\|e^{\mathcal{S}t}[\rho(\Sigma_0, F_0, r_0, q_0, p_0)] - \rho(\Sigma, F, r, q, p)\|_{H.S.} < K(t/\tau)^2 \tag{A.3}$$

where :

$$
\begin{align}
q &= q_0 + \frac{p_0}{M}t \tag{A.4} \\
p &= p_0 - 2\gamma p_0 t \tag{A.5} \\
\Sigma &= \Sigma_0 + (\frac{2}{M}\Sigma_0^2 r_0)t \tag{A.6} \\
F &= F_0 + (\frac{1}{M}\Sigma_0 F_0 r_0 - 4\gamma F_0 + \frac{2D}{\hbar})t \tag{A.7} \\
r &= r_0 - (\frac{1}{2M}\Sigma_0 r_0^2 - \frac{2\gamma}{M}F_0 - 2\gamma r_0)t \tag{A.8}
\end{align}
$$

2. Evolution under U - quadratic potential

Consider the case of the most general quadratic potential :

$$V(x) = a(x - q) + \frac{1}{2}(x - q)^2 \tag{A.9}$$

We can easily verify :

$$e^{\mathcal{U}t}[\rho(\Sigma_0, F_0, r_0, q_0, p_0)] = \rho(\Sigma_0, F_0, r_0 + bt, q_0, p_0 - at) \tag{A.10}$$

3. Evolution under U - general potential

For the case of a general potential satisfying the conditions we stated in section 4, we define :

$$W_q(x) = V(q) + V'(q) + \frac{1}{2}V^{(2)}(q)(x - q)^2 \tag{A.11}$$

Denote by $\mathcal{U}_V$ and $\mathcal{U}_{W_q}$ the generators corresponding to $V(x)$ and $W_q(x)$ respectively. We have :

$$\|e^{\mathcal{U}_V t}[\rho] - e^{\mathcal{U}_{W_q} t}[\rho]\|_{H.S}^2 \le \frac{t^2}{\hbar^2}(\rho, (\mathcal{U}_V - \mathcal{U}_{W_q})^2[\rho]) \tag{A.12}$$

in the Hilbert-Schmidt inner product. Writing : $\mathcal{O} = |U_V - U_{W_q}|$ we see that the right hand side reads :

$$\frac{t^2}{\hbar^2}(\mathcal{O}[\rho], \mathcal{O}[\rho]) = \frac{2t^2}{\hbar^2} \quad Tr(\rho^2 O^2 - \rho O \rho O) \tag{A.13}$$

where O is the operator $|V(x) - W_q(x)|$ acting on the Hilbert space H_ψ. In the coordinate representation this quantity reads :

$$\frac{2t^2}{\hbar^2} \int dx dy \rho(x,y) \rho(y,x) O(x)[O(x) - O(y)] \le$$
$$\frac{2t^2}{\hbar^2} \int dx \langle x|\rho^2|x\rangle O^2(x) =$$
$$\frac{2t^2}{\hbar^2} \frac{\Sigma}{2\pi\hbar}(\frac{\pi\hbar}{\Sigma/4 + F})^{\frac{1}{2}} \int dx \exp[-\frac{1}{\hbar}\frac{\Sigma F}{\Sigma/4 + F}(x-q)^2] O^2(x) \tag{A.14}$$

The hypothesis that $V(x)$ s uniformly Lifschitz implies that for $q \epsilon K$ (some compact subset of $\mathcal{R}$) and $|x - y| < 1$ there exists β such that:

$$|V(x) - W_q(x)| < \beta |x - y|^3$$

Following the treatment of Hagedorn we can split the integral into two sectors: $D = \{x; |x - q| \le \hbar^\alpha\}$ and $\bar{D} = \mathcal{R} - D$ where $\alpha < \frac{1}{2}$.
  The first part reads:

$$\frac{2t^2}{\hbar^2} \frac{\Sigma}{2\pi\hbar}(\frac{\pi\hbar}{\Sigma/4 + F})^{\frac{1}{2}} \int_D dx \exp[-\frac{1}{\hbar}\frac{\Sigma F}{\Sigma/4 + F}(x-q)^2] O^2(x) \le$$
$$\frac{2t^2}{\hbar^2}\beta^2 \frac{\Sigma}{2\pi\hbar}(\frac{\pi\hbar}{\Sigma/4 + F})^{\frac{1}{2}} \int_D dx \exp[-\frac{1}{\hbar}\frac{\Sigma F}{\Sigma/4 + F}(x-q)^2]|x-q|^6$$
$$\le \frac{2t^2}{\hbar^2}\beta^2(Tr\rho_0^2)^{\frac{1}{2}}\hbar^{6\alpha} \tag{A.15}$$

The second part gives :

$$\frac{2t^2}{\hbar^2}\frac{\Sigma}{2\pi\hbar}(\frac{\pi\hbar}{\Sigma/4+F})^{\frac{1}{2}}\int_{\bar{D}}dx\exp[-\frac{1}{\hbar}\frac{\Sigma F}{\Sigma/4+F}(x-q)^2]O^2(x)\leq$$

$$\frac{2t^2}{\hbar^2}C_1^2\frac{\Sigma}{2\pi\hbar}(\frac{\pi\hbar}{\Sigma/4+F})^{\frac{1}{2}}\int_{\bar{D}}dx\exp[-\frac{1}{\hbar}\frac{\Sigma F}{\Sigma/4+F}(x-q)^2]e^{2Mx^2}\leq$$

$$\frac{2t^2}{\hbar^2}C_1^2\frac{\Sigma}{2\pi\hbar}(\frac{\pi\hbar}{\Sigma/4+F})^{\frac{1}{2}}\int_{\bar{D}}dx\exp\{[-\frac{1}{2\hbar}\frac{\Sigma F}{\Sigma/4+F}+2M](x-q)^2\}$$

$$\int_{\bar{D}}dx\exp[-\frac{1}{2\hbar}\frac{\Sigma F}{\Sigma/4+F}(x-q)^2] \quad (A.16)$$

where in the last step we used the Cauchy-Schwarz inequality at $L^2(\bar{D})$ (the space of real-valued square integrable functions on $\bar{D}$). This quantity is smaller than

$$\frac{2t^2}{\hbar^2}(2Tr\rho^2)^{\frac{1}{2}}\exp[M\hbar^{2\alpha}-\hbar^{2\alpha-1}] \quad (A.17)$$

Assume that

$$(\frac{\Sigma F}{\Sigma+4F})^{-1}\leq N\hbar^{-2p} \quad (A.18)$$

for some $N>0$ and $p\leq\alpha$. This implies the existence of $\delta>0$ such that:

$$M-(\frac{\Sigma F}{\Sigma+4F})\hbar^{-1}\leq M-\hbar^{2p-1}/N^2$$

for $\hbar<\delta$ and taking $\alpha<\frac{1}{2}-p$ there exists constant c_2 such that :

$$\exp[M\hbar^{2\alpha}-\hbar^{2p+2\alpha-1}/N^2]<c_2\hbar^{6\alpha}$$

Thus we establish that when $(\frac{\Sigma F}{\Sigma+4F})^{-1}\leq N\hbar^{-2p}$, $q\epsilon K$, for each $N>0$, $p<\frac{1}{2}$, $\alpha<\frac{1}{2}-p$ and $t>0$ there exists $C>0$ and $\delta>0$, such that for $\hbar<\delta$:

$$\|e^{\mathcal{U}_V t}[\rho]-e^{\mathcal{U}_{W_q}t}[\rho]\|^2_{H.S}<C\hbar^{3\alpha-1}(Tr\rho^2)^{\frac{1}{2}}t \quad (A.19)$$

We now have expressions for the evolution induced by the generators $\mathcal{S}$ and $\mathcal{U}$. We will combine them by using the Trotter product formula.

4. Conclusion of the proof

The hypotheses on $V(x)$ ensure the existence of a bounded solution to the equations (4.2-4.6) for $t<T$. We can rewrite our previous result as :

$$\|e^{\mathcal{U}_V t}[\rho_0]-e^{\mathcal{U}_{W_q}t}[\rho_0]\|^2_{HS}<C'\hbar^\lambda/3T \quad (A.20)$$

where $\lambda = 3\alpha - 1 < \frac{1}{2}$. We discretize the system (4.2-4.7) taking time step $\delta t = t/N$. We have :

$$q_N(n) = q_0 + \sum_{i=1}^{n} \frac{p_N(i)}{M} \delta t \tag{A.21}$$

$$p_N(n) = p_0 - \sum_{i=1}^{n} [V'(q_N(i)) + 2\gamma p_N(i)] \delta t \tag{A.22}$$

$$\Sigma_N(n) = \Sigma_0 + \frac{1}{M} \sum_{i=1}^{n} [\Sigma_N^2(i) r_N(i)] \delta t \tag{A.23}$$

$$F_N(n) = F_0 + \sum_{i=1}^{n} [\frac{1}{M} \Sigma_N(i) F_N(i) r_n(i) - 4\gamma F_N(i) + \frac{2D}{\hbar}] \delta t \tag{A.24}$$

$$r_N(n) = r_0 - \sum_{i=1}^{n} [\frac{\Sigma_N(i) r_N^2(i)}{2M} + \frac{2F_N(i)}{M} + 2\gamma r_N(i)$$

$$- \frac{2}{\Sigma_N(i)} V^{(2)}(q_N(i))] \delta t \tag{A.25}$$

$$\tag{A.26}$$

Due to uniform convergence, we can always find N_1 such that for $N > N_1$:

$$\|\rho(\Sigma_N(N), F_N(N), r_N(N), q_N(N), p_N(N))$$
$$-\rho(\Sigma(t), F(t), r(t), q(t), p(t))\|_{HS} < C'\hbar^\lambda/3 \tag{A.27}$$

The Trotter product formula ensures the existence of N_2 such that for $N > N_2$:

$$\|(e^{\mathcal{L}t} - [e^{\mathcal{S}t/N} e^{\mathcal{U}t/N}]^N)[\rho_0]\|_{HS} < C'\hbar^\lambda/3 \tag{A.28}$$

For $N > max(N_1, N_2)$ we have :

$$\|[e^{\mathcal{S}t/N} e^{\mathcal{U}t/N} - e^{\mathcal{S}t/N} e^{\mathcal{W}_{q_0}t/N}][\rho(\Sigma_0, F_0, r_0, q_0, p_0)]\|_{HS} < C'\hbar^\lambda/3 \tag{A.29}$$

We also have :

$$\|e^{\mathcal{S}t/N} e^{\mathcal{U}t/N}[\rho(\Sigma_0, F_0, r_0, q_0, p_0)] - \rho(\Sigma_N(1), F_N(1), r_N(1), q_N(1), p_N(1))\|_{HS}$$
$$\leq \|[e^{\mathcal{S}t/N} e^{\mathcal{U}t/N} - e^{\mathcal{S}t/N} e^{\mathcal{W}_{q_0}t/N}][\rho(\Sigma_0, F_0, r_0, q_0, p_0)]\|_{HS}$$
$$+ \|e^{\mathcal{S}t/N} \rho(\Sigma_0, F_0, r_0 + V^{(2)}(q_0)t/N, q_0, p_0 - V'(q_0)t/N)$$
$$- \rho(\Sigma_N(1), F_N(1), r_N(1), q_N(1), p_N(1))\|_{HS}$$
$$\leq C'\hbar^\lambda/(3N) + K(t/\tau)^2 N^{-2} < C\hbar^\lambda/(3N) \tag{A.30}$$

for $N > N_3 > max(N_1, N_2)$ and with $C > C'$.

For these values of N, we can obtain by iteration :

$$\| \prod_{i=1}^{n} e^{\mathcal{S}t/N} e^{\mathcal{U}t/N} [\rho(\Sigma_0, F_0, r_0, q_0, p_0)] \quad - \quad \rho(\Sigma_N(n), F_N(n), r_N(n), q_N(n), p_N(n)\|_{HS}$$

$$\leq \quad nC\hbar^\lambda/(3N) \qquad (A.31)$$

Taking the above inequality for $n = N$ and using (9.27) and (9.30) we arrive at:

$$\|e^{\mathcal{L}t}[\rho(\Sigma_0, F_0, r_0, q_0, p_0)] - \rho(\Sigma(t), F(t), r(t), q(t), p(t))\|_{HS} < C\hbar^\lambda \quad (A.32)$$

QED